Contour-integral method for transitions to the circular unitary ensemble

2009 J. Phys. A: Math. Theor. 42 315101

(http://iopscience.iop.org/1751-8121/42/31/315101)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 315101 (11pp) doi:10.1088/1751-8113/42/31/315101

Contour-integral method for transitions to the circular unitary ensemble

Vinayak and Akhilesh Pandey

School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110 067, India

E-mail: vinayaksps2003@gmail.com and ap0700@mail.jnu.ac.in

Received 6 March 2009, in final form 15 June 2009
Published 15 July 2009
Online at stacks.iop.org/JPhysA/42/315101

Abstract
The representation of correlation functions as a contour integral has been useful in the study of transitions to the Gaussian unitary ensemble (GUE). We develop the formalism for transitions to the circular unitary ensemble (CUE) and consider the general ℓCUE to CUE transition where ℓCUE denotes a superposition of ℓ independent CUE spectra in an arbitrary ratio. For large matrices, we derive the two-level correlation function for all ℓ including $\ell = \infty$ (the Poisson case). The results are useful in the study of weakly broken partitioning symmetries and weakly coupled mesoscopic cavities.

PACS numbers: 05.45.Mt, 03.65.−w, 73.23.−b, 24.60.Lz

1. Introduction

Random-matrix theory [1–5] provides a framework for the statistical study of spectra of diverse complex quantum systems, e.g., quantum chaotic systems, mesoscopic systems, complex nuclei and atoms, etc. An important aspect of its applications is the universality of spectral fluctuations. Depending on the space–time symmetries of the system, the fluctuations are described by the three invariant random-matrix ensembles, namely, the orthogonal ensemble (OE), the unitary ensemble (UE) and the symplectic ensemble (SE). These are defined by invariance of the ensemble measure under the orthogonal, unitary and symplectic transformations respectively. Gaussian ensembles (GE) of Hermitian matrices and circular ensembles (CE) of unitary matrices are of particular interest in these studies. For GE, the invariant ensembles are GOE, GUE and GSE and for CE, the invariant ensembles are COE, CUE, CSE. For large matrices, the Gaussian and circular ensembles give the same result for the same invariance class.

For systems with a weakly broken symmetry, spectral fluctuations exhibit a transition from one universal pattern to another [6–24]. The problem of such transitions has been a subject of investigations since the 1960s when the classic papers of Rosenzweig and Porter [6] and Dyson [7] were published. In these studies, one considers a single symmetry breaking parameter τ, which governs the transition and is a relative measure of the symmetry breaking and symmetry preserving parts. For large matrices, the transition in fluctuations occurs discontinuously at $\tau = 0$. However, a smooth transition in fluctuations is obtained for small τ as a function of the appropriately rescaled transition parameter Λ [8–11]. These are verified in numerical simulations of quantum chaotic systems [3, 20–22] and have been used in the analysis of atomic [6] and nuclear [9, 13] spectra. The CE transitions are applicable to quantum chaotic maps [3, 22] and mesoscopic transport problems [4, 23, 24].

When the time reversal symmetry is broken, one observes OE–UE and SE–UE transitions [10–12]. Similarly for the breaking of a partitioning symmetry involving several quantum numbers, one considers ℓOE–OE, ℓUE–UE and ℓSE–SE transitions, where ℓ refers to the number of quantum numbers and ℓ ensembles refer to the superposition of ℓ independent spectra in arbitrary ratio [6, 9, 13–15]. For $\ell \to \infty$ the initial ensemble becomes Poisson [9, 14–19]. The transition ensembles also give identical results for the Gaussian and circular cases with suitably defined parameter Λ. For example, OE–UE and SE–UE transitions in CE [12] are found to be the same as the corresponding transitions in GE [10, 11]. Similarly transition results obtained for the Poisson to GUE [18] transition coincide with the results of Poisson to CUE transition [14, 15] and 2CUE to CUE transition results [14, 15] coincide with 2GUE to GUE results [13].

Transitions involving the Poisson ensemble as the initial condition have been studied by many authors [9, 14–19, 25–28]. However, finite ℓ is more realistic in applications. In this paper, we derive the unfolded spectral form factor for ℓCUE to CUE transition for arbitrary ℓ. Here ℓCUE is an ensemble of block-diagonal matrices with ℓ blocks of dimensions $N_1, N_2, \ldots, N_\ell$ $\left(\sum_{j=1}^{\ell} N_j = N\right)$, each block being an independent CUE. $\ell = 1$ corresponds to the case where the ensemble is CUE for all τ. On the other hand, $\ell = N$ corresponds to independent eigenangles, giving thereby the Poisson initial spectrum for $N \to \infty$. For intermediate ℓ, we have superposition of ℓ independent CUE spectra initially. These transitions apply to time-reversal noninvariant systems with a weakly broken partitioning symmetry [9, 14] or weakly coupled mesoscopic chaotic cavities [4, 23, 24]. These are analogous to transitions in time-reversal invariant systems involving, e.g., LS-breaking in atomic spectra [6], isospin breaking in nuclear spectra [13], parity breaking in quantum kicked rotors [22] and chaos transition in anisotropic Kepler problem [20].

Brezin and Hikami [25] have developed the contour-integral method to study Poisson to GUE transition. This method has been used by Kunz and Shapiro [18] to derive the two-level correlation function, given earlier by one of the present authors [14]. In this paper, we develop the contour-integral method for the circular ensembles and study transitions for all ℓ. We believe that the method is generalizable and the result is applicable to other ensembles, e.g., nonuniform circular ensembles [29] and also Laguerre and Jacobi ensembles [30].

2. The ℓCUE–CUE transition

We consider an ensemble of N-dimensional unitary matrices $U(\tau)$ which depends on the symmetry breaking parameter τ. Without loss of generality, we take the symmetry preserving part $U(0)$ to be a diagonal matrix with matrix elements $U_{jk}(0) = \exp[\mathrm{i}\phi_j]\delta_{jk}$ where ϕ_j are the eigenangles of $U(0)$. Transitions in unitary ensembles are defined in terms of the Brownian motion model [7, 12],

$$U(\tau + \delta\tau) = U(\tau)\exp[\mathrm{i}\sqrt{\delta\tau}M(\tau)]. \tag{1}$$

Here, $\delta\tau$ is infinitesimal and $M(\tau)$, independent for each τ, is a member of GUE with variance $v^2 = 1$ for real and imaginary parts of the off-diagonal matrix elements. (The corresponding

Gaussian ensembles are the ensembles of Hermitian matrices $H(\tau+\delta\tau) = H(\tau)+\sqrt{\delta\tau}M(\tau)$.) Let θ_j be the eigenangles of $U(\tau)$. We write the sets $\{\phi_1,\ldots,\phi_N\}$ and $\{\theta_1,\ldots,\theta_N\}$ as Φ and Θ respectively. Similarly we write the sets $\{e^{i\phi_1},\ldots,e^{i\phi_N}\}$ and $\{e^{i\theta_1},\ldots,e^{i\theta_N}\}$ as $e^{i\Phi}$ and $e^{i\Theta}$, and define the corresponding infinitesimal volume in N-dimensional space as $d\Theta$ and $d\Phi$. Then the joint-probability density (jpd) for transitions to CUE for arbitrary τ is given as

$$P(\Theta;\tau) = \int d\Phi\, P(\Theta,\Phi;\tau)P(\Phi;0), \tag{2}$$

where $P(\Phi;0)$ is the initial jpd. $P(\Theta,\Phi;\tau)$ is the conditional jpd given [12] by

$$P(\Theta,\Phi;\tau) = \frac{1}{N!}\frac{Q_N(\Theta)}{Q_N(\Phi)}\exp\left(\frac{N(N^2-1)\tau}{12}\right)\det[f(\theta_j-\phi_k;\tau)]_{j,k=1,\ldots,N}. \tag{3}$$

Here μ takes integral or half-integral values for $N=$ odd or even, respectively,

$$f(\psi) = \frac{1}{2\pi}\sum_{\mu=-\infty}^{\infty} e^{-\mu^2\tau+i\mu\psi} \tag{4}$$

and

$$Q_N(\Theta) = \prod_{1\leqslant k<j\leqslant N}\sin\left(\frac{\theta_j-\theta_k}{2}\right). \tag{5}$$

Note that $Q_N(\Theta)$ is antisymmetric and related to the Vandermonde determinant of variables $e^{i\Theta}$ [1]. For $\tau\to\infty$, $P(\Theta,\Phi;\infty)$ becomes independent of Φ and gives the CUE jpd

$$P(\Theta;\infty) = (2\pi)^{-N}(N!)^{-1}2^{N(N-1)}|Q_N(\Theta)|^2. \tag{6}$$

Equation (3) is equivalent to the Itzykson–Zuber integral, used in the Gaussian case [1, 10, 18]. For ℓCUE to CUE transition, the initial jpd is

$$P(\Phi;0) \propto \left[\left|Q_{N_1}(\phi_1,\ldots,\phi_{N_1})Q_{N_2}(\phi_{N_1+1},\ldots,\phi_{N_1+N_2})\cdots Q_{N_\ell}(\phi_{N-N_\ell+1},\ldots,\phi_N)\right|^2 + \text{permutations}\right]. \tag{7}$$

For $\ell=N$, $P(\Phi;0)=(1/2\pi)^N$.

We consider the ensemble average of a symmetric function $\mathcal{F}(\Theta)$ in two steps as in [14, 15, 18]. We use a bar to denote the averaging over θ_j with respect to the conditional jpd. Thus we write

$$\overline{\mathcal{F}}(\Phi) = \int d\Theta\,\mathcal{F}(\Theta)P(\Theta,\Phi;\tau), \tag{8}$$

where $\det[f(\theta_j-\phi_k)]$ in $P(\Theta,\Phi;\tau)$ can be replaced by $N!\prod_{j=1}^N f(\theta_j-\phi_j)$ using the symmetry of $\mathcal{F}$. We use angular brackets to represent the averaging over ϕ_j,

$$\langle\mathcal{F}\rangle = \int d\Phi\,\mathcal{F}(\Phi)P(\Phi;0). \tag{9}$$

Thus $\langle\overline{\mathcal{F}}\rangle$ is the average of $\mathcal{F}(\Theta)$ with respect to $P(\Theta;\tau)$.

3. Correlation functions as contour integrals

Following [18, 25] we compute the ensemble average of the following quantities

$$C_1(p) = \sum_{j=1}^{N}\exp(ip\theta_j), \tag{10}$$

$$C_2(p,q) = \sum_{\substack{j,k=1\\(j\neq k)}}^{N} \exp(\mathrm{i}p\theta_j + \mathrm{i}q\theta_k), \tag{11}$$

where p and q can take all possible integral values. Then $\langle\overline{C}_1(p)\rangle$ and $\langle\overline{C}_2(p,q)\rangle$ are the Fourier transforms of the one-point and two-point correlation functions, respectively. For integer b_j's, we have

$$\mathrm{e}^{N(N^2-1)\tau/12}\int \mathrm{d}\Theta\, Q_N(\Theta)\prod_{j=1}^{N}(f(\theta_j-\phi_j)\,\mathrm{e}^{\mathrm{i}b_j\theta_j}) = \exp\left[\sum_{j=1}^{N}\left(-\tau b_j^2 + \mathrm{i}b_j\phi_j\right)\right] Q_N(\Phi + 2\mathrm{i}\tau b), \tag{12}$$

where $\Phi + 2\mathrm{i}\tau b$ represents the set $\{\phi_1 + 2\mathrm{i}\tau b_1, \ldots, \phi_N + 2\mathrm{i}\tau b_N\}$. Equation (12) is proved in two steps. First, (8) with $\mathcal{F} = 1$ gives (12) for $b_j = 0$. Then for $b_j \neq 0$ we use the identity $\sum_{l=-\infty}^{\infty} g(l) = \sum_{l=-\infty}^{\infty} g(l+b)$ for integer l. Using this for $C_1(p)$ and $C_2(p,q)$ we get the conditional averages $\overline{C}_1(p), \overline{C}_2(p,q)$,

$$\overline{C}_1(p) = \exp[-p^2\tau - (N-1)p\tau]\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}p\phi_j}\prod_{\substack{k=1\\(k\neq j)}}^{N}\left(1 + \frac{\mathrm{e}^{\mathrm{i}\phi_k}\chi_p}{\mathrm{e}^{\mathrm{i}\phi_k} - \mathrm{e}^{\mathrm{i}\phi_j}}\right), \tag{13}$$

$$\overline{C}_2(p,q) = \exp[-p^2\tau - (N-1)p\tau]\exp[-q^2\tau - (N-1)q\tau]\sum_{\substack{j,k=1\\(j\neq k)}}^{N}\mathrm{e}^{\mathrm{i}p\phi_j + \mathrm{i}q\phi_k}F(\mathrm{e}^{\mathrm{i}\phi_j}, \mathrm{e}^{\mathrm{i}\phi_k}) \times \prod_{\substack{l=1\\(l\neq j)}}^{N}\left(1 + \frac{\mathrm{e}^{\mathrm{i}\phi_l}\chi_p}{\mathrm{e}^{\mathrm{i}\phi_l} - \mathrm{e}^{\mathrm{i}\phi_j}}\right)\prod_{\substack{l'=1\\(l'\neq k)}}^{N}\left(1 + \frac{\mathrm{e}^{\mathrm{i}\phi_{l'}}\chi_q}{\mathrm{e}^{\mathrm{i}\phi_{l'}} - \mathrm{e}^{\mathrm{i}\phi_k}}\right), \tag{14}$$

where

$$\chi_p = \exp(2p\tau) - 1 \tag{15}$$

and

$$F(z_1, z_2) = \frac{(z_1 - z_2)(z_1\,\mathrm{e}^{2q\tau} - z_2\,\mathrm{e}^{2p\tau})}{(z_1 - z_2\,\mathrm{e}^{2p\tau})(z_1\,\mathrm{e}^{2q\tau} - z_2)}. \tag{16}$$

Finally, averaging over the initial jpd we obtain

$$\langle\overline{C}_1(p)\rangle = K(p;\tau)\oint_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{z^p}{z}\prod_{j=1}^{\ell}\left\langle\prod_{k=1}^{N_j}\xi(\mathrm{e}^{\mathrm{i}\phi_k}, z, p)\right\rangle, \tag{17}$$

$$\langle\overline{C}_2(p,q)\rangle = K(p;\tau)K(q;\tau)\oint_\Gamma\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}\oint_\Gamma\frac{\mathrm{d}z_2}{2\pi\mathrm{i}}\frac{z_1^p}{z_1}\frac{z_2^q}{z_2}F(z_1,z_2)\prod_{j=1}^{\ell}\mathbf{D}_{N_j}(z_1,z_2). \tag{18}$$

Here

$$\mathbf{D}_M = \left\langle\prod_{k=1}^{M}\xi(\mathrm{e}^{\mathrm{i}\phi_k}, z_1, p)\xi(\mathrm{e}^{\mathrm{i}\phi_k}, z_2, q)\right\rangle, \tag{19}$$

$$K(p;\tau) = -\exp[-p^2\tau - (N-1)p\tau](\chi_p)^{-1}, \tag{20}$$

$$\xi(\mathrm{e}^{\mathrm{i}\phi}, z, p) = 1 + \chi_p\,\mathrm{e}^{\mathrm{i}\phi}(\mathrm{e}^{\mathrm{i}\phi} - z)^{-1}, \tag{21}$$

and contour Γ consists of two concentric circles Γ_1 and Γ_2 of radii $1+\epsilon$ and $1-\epsilon$ respectively, for small $\epsilon > 0$, enclosing all the initial eigenvalues. We choose Γ_1 and Γ_2 both in the anti-clockwise direction so that the Γ integral is the difference of the Γ_1 and Γ_2 integrals. We avoid singularities of F by choosing $|p|\tau > \epsilon$ and $|q|\tau > \epsilon$. In (19), angular brackets denote average over single M-dimensional CUE (i.e., average with respect to the jpd (6) with $N = M$). Similarly, on the right-hand side of (17) the average is over M-dimensional CUEs where $M = N_1, N_2, \ldots, N_\ell$; the same remark applies to (32). Equations (8)–(18) are analogous to corresponding equations for the Poisson to GUE transition in [18, 25].

4. The spectral form factor for large N

The conditional jpd (3) with the choice (7) has circular symmetry for the Θ spectrum. This implies stationarity of the correlation functions. Thus, for example, $\langle\overline{C_1}(p)\rangle = N\delta_{p0}$, implying constant level density. Similarly $\langle\overline{C_2}(p,q)\rangle$ is nonzero only for $p = -q$. We define the covariance

$$C(p,q) = \frac{1}{N}[\langle\overline{C_2}(p,q)\rangle - \langle\overline{C_1}(p)\rangle\langle\overline{C_1}(q)\rangle], \tag{22}$$

which is again nonzero only for $p = -q$. For large N, the unfolded spectral form factor $b_2(\mathbf{k};\Lambda)$ is defined as

$$b_2(\mathbf{k};\Lambda) = -\lim C(p,-p), \tag{23}$$

where we have taken $p = \mathrm{Int}(N\mathbf{k} + 1/2)$ and Λ is defined below after (24). The limit in (23) is for $N \to \infty$, $p \to \infty$ such that $p/N = \mathbf{k}$. In the present case, we can replace C by $\langle\overline{C_2}\rangle/N$ for $|\mathbf{k}| > (2N)^{-1}$. The two-level cluster function $Y_2(r;\Lambda)$ is given by

$$Y_2(r;\Lambda) = \int \mathrm{d}\mathbf{k}\,\mathrm{e}^{-2\pi\mathrm{i}r\mathbf{k}} b_2(\mathbf{k};\Lambda). \tag{24}$$

Here $r = (\theta_1 - \theta_2)/D$, D is the average spacing and $\Lambda = \tau v^2/D^2$ is the rescaled transition parameter [8, 12]. For CE, $D = 2\pi/N$ so that $\Lambda = \tau N^2/4\pi^2$ with our above choice $v^2 = 1$. Note that $\Lambda = O(1)$ for $\tau = O(N^{-2})$. (For GE [8, 9] with a fixed spectral span for large N, $\Lambda = O(1)$ for $\tau = O(N^{-1})$ as in figure 1.)

To calculate $\langle\overline{C_2}(p,q)\rangle$ we use the antisymmetric form of $Q_N(\Phi)$ to replace $2^{N(N-1)}|Q_N(\Phi)|^2$ by $N!\det[\exp(\mathrm{i}(l-m)\phi_m)]$ in the ϕ integrals in (19). This comes about because out of the two determinants in $|Q_N(\Phi)|^2$, each term in one of the determinants contributes equally to the integral. Thus $\mathbf{D}_M$ can be written as

$$\mathbf{D}_M = \det[\mathcal{D}_{lm}]_{l,m=1,\ldots,M}, \tag{25}$$

where

$$\mathcal{D}_{lm}(z_1,z_2) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \exp(\mathrm{i}(l-m)\phi)\xi(\mathrm{e}^{\mathrm{i}\phi}, z_1, p)\xi(\mathrm{e}^{\mathrm{i}\phi}, z_2, q). \tag{26}$$

To evaluate matrix elements $\mathcal{D}_{lm}$, note first that for integer n,

$$\int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi}\left(\frac{\mathrm{e}^{\mathrm{i}(n+1)\phi}}{\mathrm{e}^{\mathrm{i}\phi} - z}\right) = \begin{cases} z^n \mathsf{S}(n), & |z| < 1, \\ z^n[\mathsf{S}(n) - 1], & |z| > 1, \end{cases} \tag{27}$$

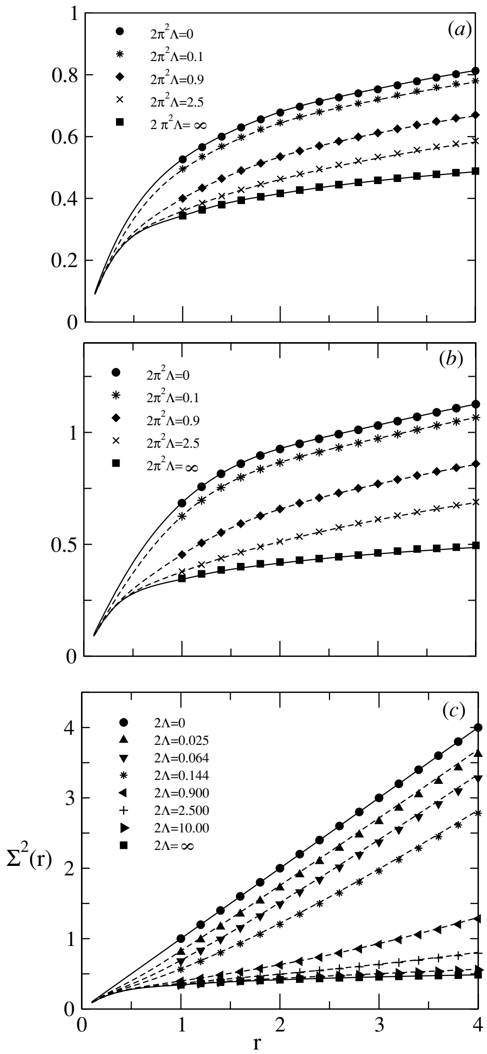

Figure 1. $\Sigma^2(r)$ versus r for: (*a*) 2UE to UE, (*b*) 3UE to UE and (*c*) Poisson to UE transitions at different Λ values. Solid lines show $\Lambda = 0, \infty$ cases. Dashed lines represent transition curves obtained by numerical integration of (45). Dots represent data from simulations of: (*a*) 2GUE to GUE with $f_1 = 0.3$ and $f_2 = 0.7$, (*b*) 3GUE to GUE with $f_1 = f_2 = f_3 = 1/3$ and (*c*) Poisson to GUE transitions. We have generated 1000-member Gaussian ensembles of 1000-dimensional matrices (999-dimensional in case (b)) and considered the middle 200 levels in each case. The initial density is semicircular in $[-2, 2]$ for (*a*) and (*b*), and uniform in [0, 1] for (*c*). With this normalization, we have $\Lambda = \tau N/2\pi^2$ in (*a*) and (*b*), and $\tau N/2$ in (*c*) for the GE. In this case $\tau = O(N^{-1})$ for $\Lambda = O(1)$.

where $\mathsf{S}(n) = 1$ for $n \geqslant 0$ and $\mathsf{S}(n) = 0$ for $n < 0$. Since the contour Γ can be split into two contours Γ_1 and Γ_2 as mentioned above, we have for a function $\mathcal{G}(z)$,

$$\oint_{\Gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \mathcal{G}(z) = \oint_{\Gamma_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \mathcal{G}(z) - \oint_{\Gamma_2} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \mathcal{G}(z). \tag{28}$$

We need to calculate matrix elements $\mathcal{D}_{lm}$ for four cases, namely, (i) $|z_1| > 1, |z_2| > 1$ (ii) $|z_1| < 1, |z_2| < 1$, (iii) $|z_1| > 1, |z_2| < 1$ and (iv) $|z_1| < 1, |z_2| > 1$. We use (27) to evaluate the integrals in (26) after making a partial-fraction expansion of the products of ξ's. It is easy to prove that for the first two cases $\mathcal{D}_{lm}$ are triangular matrices. The determinants are $\mathbf{D}_M = 1$ and $\mathbf{D}_M = \exp[2(p+q)M\tau]$ respectively for (i) and (ii). The matrix is not triangulated when z_1 and z_2 are on different circles. For (iii) we obtain, with $n = l - m$,

$$\mathcal{D}_{lm} = \delta_{n0} + \mathsf{S}(n)\big[\chi_q z_2^n + \chi_p\chi_q z_2^{n+1}(z_2 - z_1)^{-1}\big] + [\mathsf{S}(n) - 1]\big[\chi_p z_1^n - \chi_p\chi_q z_1^{n+1}(z_2 - z_1)^{-1}\big]. \tag{29}$$

To calculate $\mathbf{D}_M$, in this case, we replace the m'th column C_m by $\mathsf{C}_m - z_2\mathsf{C}_{m+1}$ for $m = 1, \ldots, M-1$ and then we replace the n'th row R_n by $\mathsf{R}_n - \mathsf{R}_{n+1}/z_1$. Now the matrix $[\mathcal{D}_{lm}]$ becomes tridiagonal and its determinant gives on expansion,

$$z_1\mathbf{D}_M = (z_1\,\mathrm{e}^{2q\tau} + z_2\,\mathrm{e}^{2p\tau})\mathbf{D}_{M-1} - z_2\,\mathrm{e}^{2(p+q)\tau}\mathbf{D}_{M-2}. \tag{30}$$

To solve the recursion relation (30), we need to calculate $\mathbf{D}_1$ and $\mathbf{D}_2$ from (29). We find $\mathbf{D}_1 = 1 + \chi_q + z_2\chi_p\chi_q(z_2 - z_1)^{-1}$. $\mathbf{D}_2$ can be determined from (30) if we choose $\mathbf{D}_0 = 1$. Solving (30) we get

$$\left(1 - \frac{z_2\,\mathrm{e}^{2p\tau}}{z_1\,\mathrm{e}^{2q\tau}}\right)\mathrm{e}^{-2qM\tau}\mathbf{D}_M = \left[\left(\frac{z_2\,\mathrm{e}^{2p\tau}}{z_1\,\mathrm{e}^{2q\tau}}\right)^{M-1} - 1\right]\frac{z_2\,\mathrm{e}^{2p\tau}}{z_1\,\mathrm{e}^{2q\tau}} + \left[1 - \left(\frac{z_2\,\mathrm{e}^{2p\tau}}{z_1\,\mathrm{e}^{2q\tau}}\right)^{M}\right]\mathrm{e}^{-2q\tau}\mathbf{D}_1. \tag{31}$$

Finally for (iv) we obtain $\mathcal{D}_{lm}$ with z_1 and p interchanged with z_2 and q respectively in (29), and then $\mathbf{D}_M$ with the same interchanges in (31). A similar calculation for the average of product of ξ's in (17) gives determinant of triangular matrices as in cases (i) and (ii). We find

$$\left\langle \prod_{k=1}^{M} \xi(\mathrm{e}^{\mathrm{i}\phi_k}, z, p)\right\rangle = \begin{cases} 1, & |z| > 1, \\ \mathrm{e}^{2pM\tau}, & |z| < 1. \end{cases} \tag{32}$$

Note that (31) and (32) contain the entire information about the initial jpd. One can use them to derive results for all N, as in [15] for $\ell = 2, \infty$. We consider here only the large-N results.

Now we derive $b_2(\mathbf{k}; \Lambda)$ for $N \to \infty$. We write $p = N\mathbf{k}$ and use the change of variables,

$$z_1 = (1 + c\delta/N)\exp[\mathrm{i}(x + y/2N)], \tag{33}$$

$$z_2 = (1 + c'\delta/N)\exp[\mathrm{i}(x - y/2N)], \tag{34}$$

in (18). Here $\delta = N\epsilon > 0$ and c, c' take values ± 1 depending on which circle z_1 and z_2 belong to. We write $f_j = N_j/N, a = 8\pi^2\Lambda, u = \mathrm{i}y + 2c\delta$ and consider large N. Then $\mathbf{D}_{N_j} = 1$ for $c = c'$ (cases (i) and (ii)). For $c \neq c'$ (cases (iii) and (iv)), we have up to $O(N^{-1}), z_1 - z_2 = u\exp(\mathrm{i}x)/N, z_1/z_2 = 1 + u/N$. Then we obtain, $\mathbf{D}_1 = 1 - a\mathbf{k}c(1 - a\mathbf{k}/u)/N, z_1^p z_2^{-p} = \exp(\mathbf{k}u), K(p;\tau)K(-p;\tau) = -N^2\exp(-a\mathbf{k}^2)/(a\mathbf{k})^2, F(z_1, z_2) = u(u - 2a\mathbf{k})/(a\mathbf{k} - u)^2$, $\mathrm{d}z_1\,\mathrm{d}z_2 = -z_1z_2\,\mathrm{d}x\,\mathrm{d}y/N$ and

$$\mathbf{D}_{N_j} = \mathrm{e}^{-a\mathbf{k}cf_j}\left[1 + \frac{(a\mathbf{k})^2(\mathrm{e}^{(2a\mathbf{k}-u)f_jc} - 1)}{u(2a\mathbf{k} - u)}\right]. \tag{35}$$

Using these large-N results, we find that the contributions of cases (i) and (ii) to $C(p, -p)$ are zero. On the other hand cases (iii) and (iv), after performing the x integral, give from (22) and (23),

$$b_2(\mathbf{k}; \Lambda) = \frac{\mathrm{e}^{-a|\mathbf{k}|(1+|\mathbf{k}|)}}{2\pi\mathrm{i}a|\mathbf{k}|}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\mathrm{d}u\,\mathrm{e}^{a\mathbf{k}^2u}\left(\frac{\prod_{j=1}^{\ell}[\mathrm{e}^{a|\mathbf{k}|f_j(2-u)} - (1-u)^2]}{(1-u)^2[u(2-u)]^{\ell-1}} + 1\right), \tag{36}$$

where we have rescaled u in (35) as $a|\mathbf{k}|u$ in (36) and taken $2\delta = a|\mathbf{k}|\gamma$. The corrections to (35) and (36) are $O(N^{-1})$. Note that the second term (+1) in the brackets of the integrand of (36) comes from $\langle\overline{C}_1(p)\rangle\langle\overline{C}_1(-p)\rangle$. We remark also that, in the intermediate step, $b_2(\mathbf{k}; \Lambda)$ is given as $\sum_c$ over the right-hand side of (36) with $|\mathbf{k}|$ replaced by $\mathbf{k}c$ everywhere including the definition of γ. But, for $\mathbf{k}c < 0$, the integral vanishes and only one of the two term survives, giving thereby (36). This is explained below.

Equation (36) is our main result. Using this result one can derive $b_2(\mathbf{k}; \Lambda)$ for all ℓCUE to CUE transitions. Note that the integrand in (36) can have poles at $u = 0, 1$ and 2. To evaluate (36), we close the contour by a vertical infinite semicircle either on the left or, along with a negative sign, on the right side of the line $\Re(u) = \gamma$ where $\gamma \to 0$. The former applies when the coefficient of u in the exponent of integrand in (36) is positive and the latter when it is negative. This ensures that the contribution of the semicircular part goes to zero for a large radius in both cases, as also does the contribution of the small rectangular strip near $\pm\mathrm{i}\infty$ in the former case. The (+1) term in (36), arising from $\langle\overline{C}_1(p)\rangle\langle\overline{C}_1(-p)\rangle$, cancels exactly the corresponding oscillatory contribution from $\langle\overline{C}_2(p, -p)\rangle$; a similar cancellation occurs in cases (i) and (ii). We have 2^ℓ terms in the expansion of (36) and for each term the location of the semicircle depends on the sign of $a|\mathbf{k}|\big(|\mathbf{k}| - \sum_{m=1}^{n} f_{\mathbf{j}_m}\big)$ where $\ell \geqslant n \geqslant 0$ and $\ell \geqslant j_m \geqslant 1$. Note also that in the intermediate step, mentioned above, with $2\delta = a\mathbf{k}c\gamma$ there is no pole enclosed by the contour for $\mathbf{k}c < 0$, since $\gamma < 0$ and sign of the coefficient of u is always positive in this case.

5. Results for $\ell = 2, 3, \infty$

For $\ell = 1$, the integrand has pole only at $u = 1$. For $|\mathbf{k}| \geqslant 1$, the contour should be closed by the semicircle on the left which contains no pole. For $|\mathbf{k}| \leqslant 1$, the contour should be closed, along with the negative sign, by the semicircle on the right which contains the pole. Thus we obtain the CUE form factor [1],

$$b_2(\mathbf{k}) = \begin{cases} 0 & \text{for} \quad |\mathbf{k}| \geqslant 1, \\ 1 - |\mathbf{k}| & \text{for} \quad |\mathbf{k}| \leqslant 1. \end{cases} \tag{37}$$

Note that the $u = 1$ pole yields (37) for all ℓ.

For $\ell = 2$ we obtain $b_2(\mathbf{k}; \Lambda)$ in four ranges of $|\mathbf{k}|$, namely, $|\mathbf{k}| \geqslant 1, 1 \geqslant |\mathbf{k}| \geqslant (f_1, f_2)_>, (f_1, f_2)_> \geqslant |\mathbf{k}| \geqslant (f_1, f_2)_<$ and $(f_1, f_2)_< \geqslant |\mathbf{k}|$. We write the result compactly as

$$b_2(\mathbf{k}; \Lambda) = b_2(\mathbf{k}; \infty) + \frac{1}{16\pi^2\Lambda|\mathbf{k}|}[\mathsf{h}(0) - \mathsf{h}(\mathbf{k}_1) - \mathsf{h}(\mathbf{k}_2) + \mathsf{h}(\mathbf{k}_3)], \tag{38}$$

where $\mathsf{h}(x) = \exp[-8\pi^2\Lambda|\mathbf{k}|(|\mathbf{k}| + 1 - 2x)]$ and $\mathbf{k}_1 = (|\mathbf{k}|, f_1)_<, \mathbf{k}_2 = (|\mathbf{k}|, f_2)_<$, and $\mathbf{k}_3 = (|\mathbf{k}|, 1)_<$. One can check that (38) can also be written as

$$b_2(|\mathbf{k}|; \Lambda) = b_2(|\mathbf{k}|; \infty) - \frac{1}{2}\left[\int_{(2|\mathbf{k}|+|f_1-f_2|,1)_>}^{2|\mathbf{k}|+1} \mathrm{d}y\, \mathsf{g}(y) - \int_{(2|\mathbf{k}|-1,1)_>}^{(2|\mathbf{k}|-|f_1-f_2|,1)_>} \mathrm{d}y\, \mathsf{g}(y)\right], \tag{39}$$

where $\mathsf{g}(y) = \exp[8\pi^2\Lambda|\mathbf{k}|(|\mathbf{k}| - y)]$. One can similarly deal with larger values of ℓ. For example, for $\ell = 3$ we obtain $b_2(\mathbf{k}; \Lambda)$ in eight ranges of $|\mathbf{k}|$,

$$\begin{aligned} b_2(\mathbf{k}; \Lambda) = b_2(\mathbf{k}; \infty) + \frac{1}{4a|\mathbf{k}|}\Bigg[&(3 - a\mathbf{k}^2)\mathsf{h}(0) - \sum_j (1 - a|\mathbf{k}|(|\mathbf{k}| + f_j - 2\mathbf{k}_j))\mathsf{h}(\mathbf{k}_j) \\ &- \sum_{j>k}(1 + a|\mathbf{k}|(|\mathbf{k}| + f_j + f_k - 2\mathbf{k}_{1+j+k}))\mathsf{h}(\mathbf{k}_{1+j+k}) \\ &+ (3 + a|\mathbf{k}|(|\mathbf{k}| + 1 - 2\mathbf{k}_7))\mathsf{h}(\mathbf{k}_7)\Bigg], \end{aligned} \tag{40}$$

where $\mathbf{k}_j = (|\mathbf{k}|, f_j)_<, \mathbf{k}_{1+j+k} = (|\mathbf{k}|, f_j + f_k)_<, \mathbf{k}_7 = (|\mathbf{k}|, 1)_<$ for $j, k = 1, 2, 3$. Integral form (39) is useful for the calculation of $Y_2(r; \Lambda)$ for $\ell = 2$. A similar integral form can be obtained from (40) for $\ell = 3$:

$$\begin{aligned} b_2(|\mathbf{k}|; \Lambda) = b_2(|\mathbf{k}|; \infty) + \frac{1}{4}\Bigg[&\sum_{j=1}^{3} \int_{(2|\mathbf{k}|-1,1)_>}^{(2|\mathbf{k}|-\alpha_j,1)_>} \mathrm{d}y\, \mathsf{g}(y) \\ &+ \sum_{j=1}^{3} \int_{(2|\mathbf{k}|-\alpha_j,1)_>}^{(2|\mathbf{k}|+\alpha_j,1)_>} \mathrm{d}y \left(1 + a|\mathbf{k}|\left\{|\mathbf{k}| - y + \frac{1-\alpha_j}{2}\right\}\right) \mathsf{g}(y) \\ &- \int_{(2|\mathbf{k}|-1,1)_>}^{2|\mathbf{k}|+1} \mathrm{d}y\, (1 + a|\mathbf{k}|(|\mathbf{k}| - y))\, \mathsf{g}(y) \\ &- \sum_{j=1}^{3} (1 + a|\mathbf{k}|\alpha_j) \int_{(2|\mathbf{k}|+\alpha_j,1)_>}^{2|\mathbf{k}|+1} \mathrm{d}y\, \mathsf{g}(y) \Bigg], \end{aligned} \tag{41}$$

where $\alpha_1 = f_2 + f_3 - f_1, \alpha_2 = f_1 + f_3 - f_2$ and $\alpha_3 = f_1 + f_2 - f_3$.

For the Poisson initial condition, $\ell \to \infty, f_j \to 0$ such that $\sum f_j = 1$. We obtain from (36),

$$b_2(\mathbf{k}; \Lambda) = \frac{\mathrm{e}^{-a\mathbf{k}^2 - a|\mathbf{k}|}}{2\pi \mathrm{i} a|\mathbf{k}|} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \mathrm{d}u\, \mathrm{e}^{a\mathbf{k}^2 u} \left[\frac{u(2-u)}{(1-u)^2} \mathrm{e}^{a\mathbf{k}/u} + 1 \right]. \tag{42}$$

In this case the infinite semicircle should be chosen in the left half plane. Then only $u = 0$ pole is enclosed by the contour which we replace by a circular contour of radius less than unity. Next the substitution $u = 1/z$ and a partial integration gives,

$$b_2(\mathbf{k}; \Lambda) = \oint_{|z|>1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{1}{z(z-1)} \left(1 - \frac{|\mathbf{k}|}{z^2}\right) \times \exp[-a|\mathbf{k}|(1 - z + |\mathbf{k}|(1 - 1/z))]. \tag{43}$$

Now, as in [18], we choose the contour $|z| = \sqrt{|\mathbf{k}|}$ in (43). In this case, the contribution of the $z = 1$ pole gets excluded for $|\mathbf{k}| < 1$ and therefore its contribution should be calculated separately. The latter gives $1 - |\mathbf{k}|$ and thus $b_2(\mathbf{k}; \infty)$ given in (37). Then the substitutions $z = \sqrt{|\mathbf{k}|} \exp(\mathrm{i}\theta)$ and $y = -\cos\theta$ give,

$$b_2(\mathbf{k}; \Lambda) = b_2(\mathbf{k}; \infty) - \frac{2}{\pi} \int_{-1}^{1} \mathrm{d}y \frac{\sqrt{1-y^2}(2y\sqrt{|\mathbf{k}|} + 1)}{|\mathbf{k}| + 2y\sqrt{|\mathbf{k}|} + 1} \times \exp[-8\pi^2 \Lambda |\mathbf{k}|(|\mathbf{k}| + 2y\sqrt{|\mathbf{k}|} + 1)]. \tag{44}$$

Results, (39) and (44), were given earlier [14, 15]. Also, (44) is obtained for Poisson to GUE transition as in [18] where $\mathbf{k}$ and 2Λ are given as $u/2\pi$ and Λ^2 respectively.

$Y_2(r; \Lambda)$ for $\ell = 2, \infty$ are given in [14]. One can similarly obtain $Y_2(r; \Lambda)$ for $\ell = 3$. We have calculated numerically the number variance [1, 2] $\Sigma^2(r; \Lambda)$,

$$\begin{aligned} \Sigma^2(r; \Lambda) &= r - \int_{-r}^{r} \mathrm{d}s (r - s) Y_2(s; \Lambda) \\ &= \int_{-\infty}^{\infty} \mathrm{d}\mathbf{k} (1 - b_2(\mathbf{k}; \Lambda)) \frac{\sin^2(\pi \mathbf{k} r)}{\pi^2 \mathbf{k}^2}. \end{aligned} \tag{45}$$

Figure 1 shows the comparison of CE and GE results for $\ell = 2, 3, \infty$. Transition occurs for $\tau = O(1/N)$ in the GE, as opposed to $\tau = O(1/N^2)$ in the CE.

6. Conclusion

By using the contour-integral method, we have obtained a compact expression for the spectral form factor for the ℓCUE to CUE transition. We believe that the same result will be obtained for ℓGUE to GUE transitions as well as for similar transitions in other ensembles [29, 30] in terms of a suitably defined parameter Λ. We can also show that the Poisson to CUE result is valid for arbitrary initial density as in [18] for Poisson to GUE; this is discussed elsewhere [19]. Our results are applicable to weakly broken partitioning symmetries in complex quantum systems with several overlapping representations and weakly coupled cavities. Finally, we mention that the original problem of Poisson to GOE and the related ℓGOE to GOE transitions, as also the corresponding COE transitions, is still largely unsolved. However there are approximate results given in [14] and for the Poisson to GOE transition some exact results are given in terms of Grassmann integrals by Guhr and Kohler [26, 27] and Datta and Kunz [28].

Acknowledgments

We thank the referees for valuable suggestions. One of us (Vinayak) acknowledges UGC, India for financial support.

References

[1] Mehta M L 2004 *Random Matrices* (New York: Academic)
[2] Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 Random-matrix physics: spectrum and strength fluctuations *Rev. Mod. Phys.* **53** 385
[3] Haake F 1991 *Quantum Signatures of Chaos* (Berlin: Springer)
[4] Beenakker C W J 1997 Random-matrix theory of quantum transport *Rev. Mod. Phys.* **69** 731
[5] Guhr T, Groeling A M and Widenmüller H A 1998 Random-matrix theories in quantum physics: common concepts *Phys. Rep.* **299** 189
[6] Rosenzweig N and Porter C E 1960 Repulsion of energy levels in complex atomic spectra *Phys. Rev.* **120** 1698
[7] Dyson F J 1962 Brownian-motion model for the eigenvalues of a random matrix *J. Math. Phys.* **3** 1191
[8] Pandey A 1981 Statistical properties of many-particle spectra: IV. New ensembles by Stieltjes transform method *Ann. Phys., NY* **134** 110
[9] French J B, Kota V K B, Pandey A and Tomsovic S 1988 Statistical properties of many-particle spectra: V. Fluctuations and symmetries *Ann. Phys., NY* **181** 198
[10] Pandey A and Mehta M L 1983 Gaussian ensembles of random Hermitian matrices intermediate between orthogonal and unitary ones *Commun. Math. Phys.* **87** 449
[11] Mehta M L and Pandey A 1983 On some Gaussian ensembles of Hermitian matrices *J. Phys. A: Math. Gen.* **16** 2655
[12] Pandey A and Shukla P 1991 Eigenvalue correlations in the circular ensembles *J. Phys. A: Math. Gen.* **24** 3907
[13] Guhr T and Weidenmüller H A 1990 Isospin mixing and spectral fluctuation properties *Ann. Phys., NY* **199** 412
[14] Pandey A 1995 Brownian-motion model of discrete spectra *Chaos Solitons Fractals* **5** 1275
[15] Pandey A 2004 Brownian-motion model of quantum-chaotic spectra: exact two-level correlations for transitions to CUE *Phase Transit.* **77** 835
[16] Guhr T 1996 Transition from Poisson regularity to chaos in a time-reversal noninvariant system *Phys. Rev. Lett.* **76** 2258
[17] Guhr T 1996 Transitions toward quantum chaos: with supersymmetry from Poisson to Gauss *Ann. Phys., NY* **250** 145
[18] Kunz H and Shapiro B 1998 Transition from Poisson to Gaussian unitary statistics: the two-point correlation function *Phys. Rev.* E **58** 400
[19] Vinayak and Pandey A 2009 Transition from Poisson to circular unitary ensemble *Pramana J. Phys.* at press
[20] Wintgen D and Marxer H 1988 Level statistics of a quantized cantori system *Phys. Rev. Lett.* **60** 971
[21] Dupuis N and Montambeaux G 1991 Aharonov–Bohm flux and statistics of energy levels in metals *Phys. Rev.* B **43** 14390

[22] Shukla P and Pandey A 1997 The effect of symmetry-breaking in 'chaotic' spectral correlations *Nonlinearity* **10** 979
[23] Frahm K and Pichard J-L 1995 Parametric statistics of the scattering matrix: from metallic to insulating quasi-unidimensional disordered systems *J. Physique* I **5** 847
[24] Frahm K and Pichard J-L 1995 Brownian motion ensembles and parametric correlations of the transmission eigenvalues: application to coupled quantum billiards and to disordered wires *J. Physique* I **5** 877
[25] Brezin E and Hikami S 1996 Correlations of nearby levels induced by a random potential *Nucl. Phys.* B **479** 697
[26] Guhr T and Kohler H 2002 Recursive construction for a class of radial functions: I. Ordinary space *J. Math. Phys.* **43** 2707
[27] Guhr T and Kohler H 2002 Recursive construction for a class of radial functions: II. Superspace *J. Math. Phys.* **43** 2741
[28] Datta N and Kunz H 2004 Random matrix approach to the crossover from Wigner to Poisson statistics of energy levels *J. Math. Phys.* **45** 870
[29] Kumar S and Pandey A 2008 Nonuniform circular ensembles *Phys. Rev.* E **78** 026204
[30] Kumar S and Pandey A 2009 Universal spectral correlations in orthogonal-unitary and symplectic-unitary crossover ensembles of random matrices *Phys. Rev.* E **79** 026211